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Constructions of infinite graphs with Ramsey property

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Abstract
For every infinite cardinal λ and $2 \leq n < \omega$ there is a directed graph D of size λ such that D does not contain directed circuits of length $\leq n$ and if its vertices are colored with $<\lambda$ colors, then there is a monochromatic directed circuit of length $n+1$. For every infinite cardinal λ and finite graph X there is a λ-sized graph Y such that if the vertices of Y are colored with $<\lambda$ colors, then there is a monochromatic induced copy of X. Further, Y does not contain larger cliques or shorter odd circuits than X. The constructions are using variants of Specker-type graphs.

KEYWORDS
directed graphs, infinite graphs, Ramsey property, Specker type graphs, undirected graphs

1 | INTRODUCTION

A. Joó proved that if $n \geq 2$, κ is an infinite cardinal, then there is a directed graph (V, D) with no directed circuit of length at most n, such that if V is colored with κ colors, then there is a monochromatic directed circuit of length $n+1$ [3]. The cardinality of V is 2^κ and Joó asked if there is a similar example with $|V| = \kappa^+$. In this note we give such an example using a variant of Specker-type graphs.

With a slight modification of the method we prove another result. If X is a finite graph and λ is an infinite cardinal, then there is a graph Y such that $|Y| = \lambda$ and $Y \Rightarrow (X)^1_{<\lambda}$. That is, if $\gamma < \lambda$ and the vertices of Y are colored with γ colors, then there is a monocolored, induced copy of X. Further, Y does not contain larger cliques or shorter odd circuits than X. If X does not contain odd circuits, that is, if it is bipartite, then for every $1 \leq s < \omega$ we can construct a Y with

$Y \Rightarrow (X)^1_{<\lambda}$ and oddgirth$(Y) = 2s + 1$ by applying the same result to the vertex disjoint union of X and a circuit of length $2s + 1$.

The same method gives the existence of a hypergraph $\mathcal{H} \subseteq [\lambda]^n$ such that the chromatic number of $\mathcal{H}$ is λ and if $H \neq H' \in \mathcal{H}$, $|H \cap H'| \geq 2$, then the elements of $H \cap H'$ are in the same positions in H and H', that is, if $y \in H \cap H'$, then $y = x_i^H = x_i^{H'}$ for some $i < n$, where $H = \{x_j^H : j < n\}_<$, $H' = \{x_j^{H'} : j < n\}_<$ are the monotonic enumerations of H, H'.

The fact that if X is a finite graph and κ is an infinite cardinal, then there exists a graph Y with $Y \Rightarrow (X)^1_\kappa$ is well known and easy to prove. One possibility is the following. If the vertex set of X is $V = \{v_0, \ldots, v_k\}$, $3 \leq k < \omega$, $\lambda = \exp_{k-1}(\kappa)^+$ we define Y as follows. The vertex set is $[\lambda]^k$, and if $A = \{a_0, \ldots, a_k\}_< \in [\lambda]^{k+1}$, then join $A - \{a_i\}$ and $A - \{a_j\}$ in Y iff $\{v_i, v_j\} \in X$. If $F : [\lambda]^k \to \kappa$ is a vertex coloring of Y, then by the Erdős-Rado theorem there is a set $S \in [\lambda]^{k+1}$ such that all k-element subsets of S get the same color. If now $S = \{s_0, \ldots, s_k\}$, $B_i = S - \{s_i\}$, then by construction $\{B_i, B_j\} \in Y$ iff $\{v_i, v_j\} \in X$, that is, $[S]^k$ gives an induced monochromatic copy of X (see [2] and [4]).

Another construction gives that if μ, κ are cardinals, $\mu \leq \kappa$, κ infinite, then for every graph X on μ, there is a graph Y with $|Y| = 2^\kappa$, with $Y \Rightarrow (X)^1_\kappa$. The vertex set of Y is ${}^\kappa\mu$, if $f, g \in {}^\kappa\mu$, then $\{f, g\} \in Y$ iff for $\delta = \min\{\alpha < \kappa : f(\alpha) \neq g(\alpha)\}$ one has $\{f(\delta), g(\delta)\} \in X$. A Baire category-type argument shows that $Y \to (X)^1_\kappa$. (A similar idea is used in Joó's above mentioned construction).

The following generalization is presented in [5]. There, if n, t are finite, κ infinite, X is an n-chromatic graph, $|X| \leq \kappa$, a graph Y is given with $Y \to (X)^1_\kappa$ such that $|Y| = \kappa^+$, and each subgraph of Y of size $<t$ is n-colorable. Notice that here only a not necessarily induced monochomatic copy of X is guaranteed.

The advantage of our construction over these examples is that it gives an induced copy and it has optimal size: κ^+ as opposed to $\exp_{k-1}(\kappa)^+$ or 2^κ.

In the definition of the directed and undirected graphs we utilize Specker graphs, that is, when the underlying set is $[\lambda]^t$ for some finite t, with $\bar{x}, \bar{y} \in [\lambda]^t$ joined if $\bar{x} \cap \bar{y} = \varnothing$ and they interlace in a predetermined way.

Specker-type graphs were introduced by Specker in [7] originally to prove the negative ordinary partition relation $\omega^3 \not\to (\omega^3, 3)^2$. Then Erdős and Hajnal used them in [1] to give examples of large chromatic graphs with arbitrarily large (finite) odd girth. Our method differs from those in [1] and [5] in that not one but finitely many interlacing possibilities are allowed.

1.1 | Notation

Definition We use the notions and notation of axiomatic set theory. In particular each cardinal is the least ordinal with that cardinality. If κ is a cardinal, $\exp(\kappa) = \exp_1(\kappa) = 2^\kappa$, and recursively $\exp_{r+1}(\kappa) = \exp(\exp_r(\kappa))$.

If f is a function, A a set, then $f[A] = \{f(x) : x \in A\}$. If S is a set, κ a cardinal, then $[S]^\kappa = \{\bar{x} \subseteq S : |\bar{x}| = \kappa\}$. Finite subsets of some set are typically denoted by $\bar{x}, \bar{y}, \ldots$. The notation $\{x_0, x_1, \ldots, x_{n-1}\}_<$ means that $x_0 < x_1 < \cdots < x_{n-1}$.

A partially ordered set $(P, <)$ is a pair where $<$ is a binary relation on P which is irreflexive, transitive but not necessarily trichotomic. If $x \in P$, then $x\downarrow = \{y \in P : y < x\}$.

A *graph* is a pair (V, X) where $X \subseteq [V]^2$. A *directed graph* is a graph in which each edge has a direction, that is, $X \subseteq V \times V$, with no loops, that is, edges of the type $\langle v, v \rangle$. If X is a graph, then $\langle v_0, v_1, \ldots, v_n \rangle$ is a *cycle*, if $\{v_i, v_{i+1}\} \in X$ for $i < n$, where formally $v_n = v_0$. Here, n is the *length of the* *cycle*. A *circuit* is a cycle with distinct vertices. oddgirth(X) is the length of the shortest circuit of odd length. Similarly, in a directed graph a *directed cycle* is $\langle v_0, ..., v_n = v_0\rangle$ where $\langle v_i, v_{i+1}\rangle \in D$ ($i < n$). A *directed circuit* is a directed cycle with distinct vertices.

A *clique* is a complete subgraph in some graph. The *clique number* is the size of the largest clique (if finite).

If X, Y are graphs, γ a cardinal, then $Y \Rightarrow (X)^1_\gamma$ denotes that whenever the vertex set of Y is colored with γ colors, then there is a monocolored induced copy of X. $Y \Rightarrow (X)^1_{<\kappa}$ denotes that $Y \Rightarrow (X)^1_\gamma$ holds for every $\gamma < \kappa$. If a not necessarily induced copy is guaranteed, then we write $Y \to (X)^1_\gamma$.

A *hypergraph* $\mathcal{H}$ on V is any set of subsets of V, where $1 < |H| < \omega$ holds for $H \in \mathcal{H}$. Its *chromatic number*, $\mathrm{Chr}(\mathcal{H})$ is the least cardinal κ such that there is a *good coloring* $f: V \to \kappa$, that is, such that $|f[H]| \geq 2(H \in \mathcal{H})$.

Definition If F: $[\kappa^+]^t \to \kappa$, $\gamma < \kappa$, we let $\phi^\gamma_t(x_0, ..., x_{t-1})$ be the following formula: $x_0 < \cdots < x_{t-1} < \kappa^+$ and $F(x_0, ..., x_{t-1}) = \gamma$.

By reverse recursion on $i < t$ we define $\phi^\gamma_i(x_0, ..., x_{i-1})$ as

$$(\exists^* x_i)\phi^\gamma_{i+1}(x_0, ..., x_i),$$

where $\exists^*$ is the quantifier "there exist unboundedly many." This way, ϕ^γ_0 is the formula

$$(\exists^* x_0)(\exists^* x_1) \cdots (\exists^* x_{t-1})[(x_0 < \cdots < x_{t-1}) \wedge (F(x_0, ..., x_{t-1}) = \gamma)].$$

Theorem 1. *If* $F: [\kappa^+]^t \to \kappa$, *then* ϕ^γ_0 *holds for some* $\gamma < \kappa$.

Proof. We show that for every $s \in [\kappa^+]^{\leq t}$ there is $\gamma(s) < \kappa$ such that $\phi^{\gamma(s)}_{|s|}(s)$ holds. This we prove by reverse induction on $|s|$.

If $|s| = t$, $\gamma(s) = F(s)$ applies.

If $|s| < t$ and the statement holds for all s' with $|s'| > |s|$, for every $\max(s) < \xi < \kappa^+$ we have a $\gamma(s \cup \{\xi\})$. There is $i < \kappa$ such that $\{\xi : \gamma(s \cup \{\xi\}) = i\}$ is cofinal in κ^+. Set $\gamma(s) = i$.

The statement for $s = \varnothing$ gives the Theorem. □

Theorem 2. *If* $M < \omega$, $\mathcal{H} \subseteq [M]^t$ *is a partition of* M, $F: [\kappa^+]^t \to \kappa$, *then there is* $\gamma < \kappa$, *and an order preserving* $g: M \to \kappa^+$ *such that* $F(g[H]) = \gamma(H \in \mathcal{H})$.

Proof. Let $\gamma < \kappa$ be such that ϕ^γ_0 holds (cf. Theorem 1). We define $g(j)$ by recursion on $j < M$ such that

(a) $g(j-1) < g(j)$ $(j < M)$, and
(b) if $H = \{x^H_0, ..., x^H_{t-1}\}_<$ is the unique element of $\mathcal{H}$ containing j, $j = x^H_i$, then $\phi^\gamma_{i+1}(g(x^H_0), ..., g(x^H_i))$ holds.

The selection of $g(j)$ is possible, as by the inductive assumption the statement $\phi^\gamma_i(g(x^H_0), ..., g(x^H_{i-1}))$ holds, therefore, there exists an appropriate $g(x^H_i) > g(j-1)$.

Having finished the construction, as $\phi^\gamma_t(g(x^H_0), ..., g(x^H_{t-1}))$ holds, we have $F(g[H]) = \gamma$ $(H \in \mathcal{H})$ □

Definition If $2 \le t < \omega$, $0 \le a \le t$, $\bar{x}, \bar{y} \in [\kappa^+]^t$, then $R^a(\bar{x}, \bar{y})$, if $\bar{x} = \{x_0, \ldots, x_{t-1}\}_<$, $\bar{y} = \{y_0, \ldots, y_{t-1}\}_<$, and

$$x_a < y_0 < x_{a+1} < y_1 < x_{a+2} < y_2 < \cdots < x_{t-1} < y_{t-a-1}.$$

Lemma 3. *If $R^a(\bar{x}, \bar{y})$ and $g: \bar{x} \cup \bar{y} \to \mathrm{ORD}$ is strictly increasing, then $R^a(g[\bar{x}], g[\bar{y}])$ holds.*

Proof. As $x_i < y_j$ iff $g(x_i) < g(y_j)$, where $\bar{x} = \{x_i : i < t\}_<$ and $\bar{y} = \{y_j : j < t\}_<$. □

2 | CONSTRUCTION OF A DIRECTED GRAPH

Definition For $2 \le n < \omega$, $t = 2N + 1$, $0 < a < b = na < \omega$ with the values of N, a to be determined later, we define the directed graph D on $[\kappa^+]^t$ as follows. If $\bar{x}, \bar{y} \in [\kappa^+]^t$, $\min(\bar{x}) < \min(\bar{y})$, then $\bar{x} \leftarrow \bar{y}$ if $R^a(\bar{x}, \bar{y})$, $\bar{x} \to \bar{y}$ if $R^b(\bar{x}, \bar{y})$.

Lemma 4. *D contains a directed circuit of length $n + 1$.*

Proof. Define $\bar{x}^i = \{x^i_j : j \le 2N\}$, where $x^i_j = (ia + j)T + i$ $(0 \le i \le n,\ 0 \le j \le 2N)$ for some $T > n$. Inspection shows that $R^a(\bar{x}^i, \bar{x}^{i+1})$ $(0 \le i < n)$ and $R^b(\bar{x}^0, \bar{x}^n)$. □

Lemma 5. *Let $\bar{x} = \{x_0, \ldots, x_{2N}\}_<, \bar{y} = \{y_0, \ldots, y_{2N}\}_< \in [\kappa^+]^{2N+1}$, $R^a(\bar{x}, \bar{y})$, $x_i < c < x_{i+k}$ with $a + 1 < i$, then $y_{i-a-1} < c < y_{i+k-a}$.*

Proof. As $y_{i-a-1} < x_i < c < x_{i+k} < y_{i+k-a}$. □

Lemma 6. *Let $\bar{x} = \{x_0, \ldots, x_{2N}\}_<, \bar{y} = \{y_0, \ldots, y_{2N}\}_< \in [\kappa^+]^{2N+1}$, $R^b(\bar{y}, \bar{x})$, $x_i < c < x_{i+k}$ with $i + b + k + 1 < 2N$, then $y_{i+b} < c < y_{i+k+b+1}$.*

Proof. As $y_{i+b} < x_i < c < x_{i+k} < y_{i+b+k+1}$. □

Lemma 7. *D does not contain directed circuits of length $\le n$ (for appropriately chosen values of N and a).*

Proof. Assume for a contradiction that $\bar{x}^0, \ldots, \bar{x}^m = \bar{x}^0$ form a directed circuit of length $m \le n$. Let r, s be such that we have r times $R^a(\bar{x}^i, \bar{x}^{i+1})$ and s times $R^b(\bar{x}^{i+1}, \bar{x}^i)$. Clearly $r + s = m$. Notice that $r \ge 1$ and $s \ge 1$ as otherwise we have an increasing or decreasing directed path.

Pick $c = x^0_N$, the middle element of $\bar{x}^0$. Then obviously $x^0_{N-1} < c < x^0_{N+1}$. Going around the circuit, applications of Lemmas 5 and 6 give

$$x^0_{N-1-r(a+1)+sb} < c < x^0_{N+1-ra+s(b+1)},$$

assuming the condition for the Lemmas hold, for which $b + 1 < N - (b+1)(n-1)$, that is, $N > (b+1)n$, is more than enough.

Comparing the above two inequalities, we must have

$$N - 1 - r(a + 1) + sb \leq N - 1$$

and

$$N + 1 - ra + s(b + 1) \geq N + 1$$

that is,

$$-r(a + 1) + sb \leq 0$$

and

$$-ra + s(b + 1) \geq 0.$$

As $b = na$, we obtain $-r(a+1) + sna \leq 0 \leq -ra + s(na+1)$. This can be written as $-s \leq -ra + sna \leq r$, that is, $|(sn - r)a| \leq \max(r, s) < n$. If now $a = n$, then this can be rewritten as $|sn - r| < 1$, that is, $r = sn \geq n$, contradiction. □

Theorem 8. *If λ is an infinite cardinal, $2 \leq n < \omega$, then there is a directed graph D such that D does not contain directed circuits of length $\leq n$ and if X is colored with $<\lambda$ colors, then there is, in some color class, a directed circuit of length $n + 1$.*

Proof. The case $\lambda = \omega$ follows from the existence of finite examples (see, eg, [1]). Assume therefore, that $\lambda > \omega$. It suffices to show the result for $\lambda = \kappa^+$, as otherwise, that is, if λ is a limit cardinal, we can take the vertex disjoint union of smaller examples.

For $\lambda = \kappa^+$, we let D be the directed graph investigated above. Notice the conditions on the parameters we earlier established: $a = n$, $b = na = n^2$, $N \geq (b+1)n + 1 = (n^2 + 1)n + 1$, $t = 2N + 1$, and the directed graph is on the set $[\kappa^+]^t$.

Lemma 7 gives that there are no directed circuits of length $\leq n$. If $F : [\kappa^+]^t \to \kappa$, then, by Theorem 2, there is a copy of the directed circuit of Lemma 4 in $F^{-1}(\gamma)$ for some $\gamma < \kappa$. □

3 | CONSTRUCTION OF AN UNDIRECTED GRAPH

In the following part of the paper we prove a similar partition result for undirected graphs.

Lemma 9. *If $R^a(\bar{x}, \bar{y})$, $R^b(\bar{y}, \bar{z})$, and $R^c(\bar{x}, \bar{z})$ then $c = a + b$ or $a + b + 1$.*

Proof. As $x_{a+b} < y_b < z_0 < y_{b+1} < x_{a+b+2}$. □

Definition Let X be an undirected graph on $n=\{0,1,\ldots,n-1\}$, κ an infinite cardinal. Assume that $a_0<a_1<\cdots<a_{n-1}$, $a_i=a^{i+1}$, are natural numbers with a to be determined later, $t\geq 2a_{n-1}$. We define the graph Y on $[\kappa^+]^t$ as follows. If $\bar{x},\bar{y}\in[\kappa^+]^t$, then $\{\bar{x},\bar{y}\}\in Y$ iff there is an edge $\{i,j\}\in X$ such that $R^{|a_i-a_j|}(\bar{x},\bar{y})$ holds. Naturally, we need that the following holds: if $a_{i_0}-a_{j_0}=a_{i_1}-a_{j_1}$, then $i_0=i_1$, $j_0=j_1$. This is implied by the next Lemma.

Lemma 10. *Assume that* $a_i=a^{i+1}(i=0,\ 1,\ldots,n-1)$, k_i *is integer,* $|k_i|<a(i<n)$, $k_0a_0+k_1a_1+\cdots+k_{n-1}a_{n-1}=0$, *then* $k_0=k_1=\cdots=k_{n-1}=0$.

Proof. Assume for a contradiction that $k_0a_0+\cdots+k_{n-1}a_{n-1}=0$ yet not all of $k_0,\ldots,k_{n-1}$ are zero. Let r be least that $k_r\neq 0$. Then $k_ra_r+k_{r+1}a_{r+1}+\cdots+k_{n-1}a_{n-1}=0$. Dividing by a^{r+1} and reordering we obtain

$$k_r=-(k_{r+1}a+\cdots+k_{n-1}a^{n-r-1}).$$

Now the RHS is divisible by a, therefore, as $-a<k_r<a$, one must have $k_r=0$, contradicting the assumptions. □

Lemma 11. *Assume that* $a_i=a^{i+1}(i=0,\ldots,n-1)$, k_i *is an integer, and* $|k_0a_0+\cdots+k_{n-1}a_{n-1}|<a$, *then* $k_0a_0+k_1a_1+\cdots+k_{n-1}a_{n-1}=0$.

Proof. As $k_0a_0+\cdots+k_{n-1}a_{n-1}$ is divisible by a, and has absolute value $<a$, we have $k_0a_0+\cdots+k_{n-1}a_{n-1}=0$. □

Lemma 12. *Y contains an induced subgraph isomorphic to X.*

Proof. Set $\bar{x}^i=\{x(i,j):j<t\}$ $(i<n)$, where $x(i,j)=(a_i+j)n+i(i<n,\ j<t)$. Then $x(i_0,j_0)\neq x(i_1,j_1)$ if $\langle i_0,j_0\rangle\neq\langle i_1,j_1\rangle$. Further, $x(i_0,j_0)<x(i_1,j_1)$ iff either $i_0=i_1$ and $j_0<j_1$, or $i_0<i_1$ and $a_{i_0}+j_0\leq a_{i_1}+j_1$, that is, $(a_{i_0}-a_{i_1})+j_0\leq j_1$ or $i_1<i_0$ and $a_{i_0}-a_{i_1}<j_1-j_0$.

This implies that $R^b(\bar{x}^{i_0},\bar{x}^{i_1})$ holds $(i_0<i_1)$ iff $b=a_{i_1}-a_{i_0}$. The definition of Y gives that $\{\bar{x}^{i_0},\bar{x}^{i_1}\}\in Y$ iff $\{i_0,i_1\}\in X$. □

Lemma 13. *If $\{\bar{x},\bar{y},\bar{z}\}$ is a triangle in Y,* $\min(\bar{x})<\min(\bar{y})<\min(\bar{z})$, *$a\geq 4$, then there are i, j, k, such that $R^{a_j-a_i}(\bar{x},\bar{y})$, $R^{a_k-a_i}(\bar{x},\bar{z})$, $R^{a_k-a_j}(\bar{y},\bar{z})$, and $\{i,j,k\}$ is a triangle in X.*

Proof. By the definition of Y, if $\{\bar{x},\bar{y},\bar{z}\}$ is a triangle, then there are $\alpha>\beta$, $\gamma>\delta$, and $i>j$ such that $R^{a_\alpha-a_\beta}(\bar{x},\bar{y})$, $R^{a_\gamma-a_\delta}(\bar{y},\bar{z})$, and $R^{a_i-a_j}(\bar{x},\bar{z})$. By Lemma 9, this implies that either

$$a_\alpha-a_\beta+a_\gamma-a_\delta=a_i-a_j$$

or else

$$a_\alpha-a_\beta+a_\gamma-a_\delta+1=a_i-a_j.$$

By Lemma 11, the latter cannot hold, so we have the former.

By Lemma 10, if we reorder this to one side, all coefficients must be 0. *Case* 1. $\alpha=\gamma$.

In this case, after reordering all terms to the LHS, the coefficient of a_α is ≥ 1, a contradiction.

Case 2. $\alpha > \gamma$.

In this case, a_α is the term with the largest index on the LHS, therefore $\alpha = i$. Subtracting the common term from both sides, we obtain

$$-a_\beta + a_\gamma - a_\delta = -a_j.$$

By the property in Lemma 9, we have the following possibilites.

Case 2.1. $\beta = j$.

Then $\gamma = \delta$, impossible.

Case 2.2. $j = \delta$.

Then $a_\beta = a_\gamma$, that is, $\beta = \gamma$.

Case 3. $\alpha < \gamma$.

We proceed as in Case 2. □

Lemma 14. *If* $\bar{x}^0, \bar{x}^1, \ldots, \bar{x}^{k-1}$ *form a clique in* Y *with* $\min(\bar{x}^0) < \min(\bar{x}^1) < \cdots < \min(\bar{x}^{k-1})$ $(k \geq 3)$*, then there are distinct* $i(0), \ldots, i(k-1)$*, such that* $R^{a_{i(j_1)} - a_{i(j_0)}}(\bar{x}^{j_0}, \bar{x}^{j_1})(j_0 < j_1)$.

Here, and later, if $a < 0$, then $R^a(\bar{x}, \bar{y})$ means $R^{-a}(\bar{y}, \bar{x})$.

Proof. By induction on k. The case $k = 3$ is covered in Lemma 11

Assume we have the statement for $\bar{x}^0, \ldots, \bar{x}^{k-1}$ and $R^{a_{j_1} - a_{j_0}}(\bar{x}^{j_0}, \bar{x}^{j_1})$ for $j_0 < j_1 < k$. By Lemma 13 for any $j < j' < k$ there is $i(j, j')$ such that $R^{a_{i(j)} - a_{i(j,j')}}(\bar{x}^j, \bar{x}^k)$ and $R^{a_{i(j')} - a_{i(j,j')}}(\bar{x}^{j'}, \bar{x}^k)$. For any other j'' with $j < j'' < k$ we have $a_{i(j)} - a_{i(j,\ j')} = a_{i(j)} - a_{i(j,\ j'')}$ and so $i(j, j') = i(j, j'')$ $(j'' \neq j, j')$ and similarly $i(j, j') = i(j^*, j')$ $(j^* \neq j, j')$. This can only be if $i(j, j')$ is the same for any $j < j' < k$ and this will be $i(k)$. □

Lemma 15. *oddgirth*$(Y) =$ *oddgirth*(X).

Proof. Assume that $\langle \bar{x}^0, \bar{x}^1, \ldots, \bar{x}^{2\nu+1} = \bar{x}^0 \rangle$ is a circuit in Y of length $2\nu + 1 <$ oddgirth (X). There are $i(k) \neq j(k) < n$ such that $R^{a_{i(k)} - a_{j(k)}}(\bar{x}^k, \bar{x}^{k+1})$.

Let $t = 2N + 1$ and let c be the middle element of $\bar{x}^0$, that is, $c = x^0_N$. Then $x^0_{N-1} < c < x^0_{N+1}$ and repeated applications of Lemmas 5 and 6 give

$$x^0_{N+T-(2\nu+2)} < c < x^0_{N+T+(2\nu+2)},$$

where

$$T = (a_{i(0)} - a_{j(0)}) + (a_{i(1)} - a_{j(1)}) + \cdots + (a_{i(2\nu)} - a_{j(2\nu)}).$$

These give $|T| \leq 2\nu + 2$. If $2\nu + 2 <$ oddgirth$(X) \leq a$, then $T = 0$ by Lemma 11.

Lemma 10 gives

$$\sum_{i(u)=\ell} 1 - \sum_{j(u)=\ell} 1 = 0$$

for any $\ell < n$. For each $u \leq 2v$ take a directed edge $e(u)$ from $i(u)$ to $j(u)$. This way, we obtain a directed multigraph Z whose edges, without the direction, are edges of X. Further, $d_{\text{in}}(u) = d_{\text{out}}(u)$ holds for every $u < n$ in Z. Then Z splits into the edge-union of directed circuits in X, one of them, say C, must be of odd length. But then, the undirected version of C is an odd circuit in X, a contradiction, as $2v + 2 < \text{oddgirth}(X)$. □

Theorem 16. *If X is a finite graph, $\lambda \geq \omega$ is a cardinal, then there is a graph Y of cardinality λ with the same clique number and the same odd girth as X, and $Y \Rightarrow (X)^1_\gamma (\gamma < \lambda)$. This holds for edge-colored graphs as well: if $g: X \to r$ is a coloring of the edges of X, then there is an edge-coloring g' of Y such that if the vertices of Y are colored with $<\lambda$ colors then there is a monochromatic induced colored copy of X.*

Proof. Again, as in Theorem 8 it suffices to show the case $\lambda = \kappa^+$, $\gamma = \kappa$. (For $\lambda = \omega$, examples are given in [1]).

We have to select the parameters satisfying the following inequalities and equalities: $a \geq \max(4, \text{oddgirth}(X))$, $a_i = a^{i+1} (i < n)$, $N \geq a_n$, $t = 2N + 1$.

We define Y as in the Definition following Lemma 9. $Y \Rightarrow (X)^1_\kappa$ can be proved exactly as in the proof of Theorem 8.

The clique number and oddgirth of Y are the same as those of X by Lemmas 14 and 15.

The edge-colored version of the Theorem follows immediately from the fact that for every edge of $e \in Y$ there is a unique edge $f(e) \in X$ such that if in our arguments $g: n \to [\kappa^+]^t$ is an embedding into an induced copy of X, then $f(\{g(i), g(j)\}) = \{i, j\}$. □

Before turning to Theorem 19, we prove two Lemmas on well founded partially ordered sets.

Lemma 17. *If $(P, <)$ is a well founded, partially ordered set, then there is a well order $<_w$ of P which extends $<$, that is, if $x < y$, then $x <_w y$.*

Proof. By recursion define the elements $p(\alpha)$ as follows. If $\{p(\beta) : \beta < \alpha\} \neq P$, we let $p(\alpha)$ be a minimal element of $P - \{p(\beta) : \beta < \alpha\}$ (exists, as $(P, <)$ is well founded). If $\{p(\beta) : \beta < \alpha\} = P$ we stop the construction. Clearly, $P = \{p(\alpha) : \alpha < \phi\}$ for an appropriate ordinal ϕ. If $x = p(\beta)$, $y = p(\alpha)$, $\beta < \alpha$, set $<_w y$. Clearly, $<_w$ is a well ordering.

To check the other property, assume that $x = p(\beta)$, $y = p(\alpha)$, $x < y$, but $y <_w x$, that is, $\alpha < \beta$. This means, that when choosing $y = p(\alpha)$, x was as yet unselected, and so, as $x < y$, y was not minimal, a contradiction. □

Lemma 18. *Assume that $(P, <)$ is a well founded partially ordered set, $|P| = \lambda$, λ is regular, and $x\downarrow$ has cardinality $<\lambda$ for $x \in P$. Then $<$ can be extended to a well order of type λ.*

Proof. Enumerate P as $\{p_\alpha : \alpha < \lambda\}$. Define

$$P_\alpha = \{x : \exists \beta < \alpha [x \leq p_\beta]\}$$

for $\alpha < \lambda$. Then $\langle P_\alpha : \alpha < \lambda \rangle$ is an increasing, continuous sequence of subsets with $\bigcup \{P_\alpha : \alpha < \lambda\} = P$, $P_0 = \emptyset$, $|P_\alpha| < \lambda (\alpha < \lambda)$. Let $<_\alpha$ be a well ordering of $P_{\alpha+1} - P_\alpha$ extending $<$ (Lemma 17).

Define $<_w$ on P as follows. If $x \neq y \in P$, $x \in P_{\alpha+1} - P_\alpha$, $y \in P_{\beta+1} - P_\beta$, then set $x <_w y$ if either $\alpha < \beta$ or else $\alpha = \beta$ and $x <_\alpha y$.

We show that $<_w$ is as required. It is clearly a well ordering. Assume that $x < y$. Let α be the ordinal such that $y \in P_{\alpha+1} - P_\alpha$. By construction, $x \in P_{\alpha+1}$. If $x \in P_\alpha$, then $x <_w y$. If $x \in P_{\alpha+1} - P_\alpha$, then $x <_\alpha y$ and so again $x <_w y$.

Finally, if $x \in P_\alpha$, then the initial segment of $<_w$ determined by x is a subset of P_α and $|P_\alpha| < \lambda$, so each proper segment is of size $<\lambda$, consequently the type of $<_w$ is at most λ. As $|P| = \lambda$, it is equal to λ. □

Theorem 19. *If $\lambda > \omega$ is a cardinal, $3 \leq n < \omega$, then there is a family $\mathcal{H} \subseteq [\lambda]^n$ such that*

(a) if $H = \{y_0^H, \ldots, y_{n-1}^H\}_<$ for $H \in \mathcal{H}$, $H, H' \in \mathcal{H}$ are such that $|H \cap H'| \geq 2$, $z \in H \cap H'$, then $z = y_i^H = y_i^{H'}$ for some $i < n$, and

(b) if $F : \lambda \to \gamma$ for some $\gamma < \lambda$, then there is an $H \in \mathcal{H}$ monocolored by F.

Proof. Again, we can assume that $\lambda = \kappa^+$.

Let t, $a_0, \ldots, a_{n-1}$ be as in the above definitions. We set $H \in \mathcal{H}$ if $H = \{\bar{x}^0, \ldots, \bar{x}^{n-1}\}$ for some t-tuples such that $R^{a_j - a_i}(\bar{x}^i, \bar{x}^j)$ holds for $i < j < n$.

We have the properties required, the only problem is that the underlying set is $[\kappa^+]^t$. Define a relation $<$ on $[\kappa^+]^t$ by $\bar{x} < \bar{y}$ if $R^{a_j - a_i}(\bar{x}, \bar{y})$ holds for some $i < j < n$. The transitive closure of it is a partial ordering, again denoted by $<$. It is well founded, as $\bar{x} < \bar{y}$ implies $\min(\bar{x}) < \min(\bar{y})$. Further, if $\bar{x} < \bar{y}$, then $\bar{x} \in [\max(\bar{y})]^t$, whose cardinality is at most κ.

Using Lemma 18, we can find a well order $<_w$ of $[\kappa^+]^t$ whose order type is κ^+ and under which each $H = \{\bar{x}^0, \ldots, \bar{x}^{n-1}\}$ has $\bar{x}^0 <_w \cdots <_w \bar{x}^{n-1}$, and so the result is proved. □

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